2022/2023學年學界數學比賽　高中組

Concurso Interescolar de Matemática do ano lectivo de 2022/2023　Divisão Sênior

Macau Interschool Mathematics Competition in 2023/2023　Senior Division

學校 School ______________ 姓名 Name ______________ 座號 Seat No.________

- 比賽共有12題：題1 – 5是填空題，把答案填寫在右方空格內。題6 – 12是證明題，請把詳細的解題過程寫在背後的白紙上，並在左上角寫上題目編號及姓名。題1 – 7 每題10 分，題8 – 12每題20 分。禁止使用任何類型之計算機或計算工具。

- There are 12 questions. Questions 1–5 are fill-in-the-blanks, write the answers in the box on the right. Questions 6–12 are proof-related, please write the arguments/steps of your solutions at the blank page with question number and your name on the left-upper corner. Each of questions 1–7 is 10 marks, and each of questions 8–12 is 20 marks. Any electronic calculators or computing tools of any kinds are not allowed

符號　Notations

1. 符號$|x|$ 表示實數x的絕對值。例如: $|-13|=13$、$|0|=0$、$|13|=13$。
   The symbol $|x|$ represents the absolute value of real number x.
   For example, $|-13|=13$, $|0|=0$, $|13|=13$.

2. 設a,b爲整數且$a\neq 0$, 符號$a\mid b$ 表示b是a的倍數, 即有某個整數q使得$b=aq$ 。
   Let a,b be two integers and $a\neq 0$, the notation $a\mid b$ means that b is a multiple of a, i.e. $b=aq$ for some integer q.

3. $N=\left\lfloor \frac{a}{k}\right\rfloor$ 代表分數$\frac{a}{k}$的整數部份, 即N 是滿足不等式$n\leq\frac{a}{k}$的所有整數n中的最大值。
   $N=\left\lfloor \frac{a}{k}\right\rfloor$ represents the integral part of the fraction $\frac{a}{k}$, that is, among all integers n satisfying the inequality $n\leq\frac{a}{k}$, N is the largest one.

4. 設k爲整數，稱 a 爲k 的正約數 若a 滿足以下兩個條件：(i) a 是正整數及 (ii) k是a的倍數。
   Let k be an integer, a is called a positive divisor of k, if a satisfies the following two conditions: (i) a is a positive integer and (ii) k is a multiple of a.

5. 符號 $\sigma(k)$ 代表正整數k的所有正約數之和。
   The symbol $\sigma(k)$ represents the sum of all positive divisors of positive integer k.

6. 符號$\varphi(k)$表示歐拉函數在正整數k的取值，即$\varphi(k)$表示在首k個正整數中與k互素的整數之個數。
   The symbol $\varphi(k)$ represents the value of the Euler-totient function evaluated at k, that is $\varphi(k)$ is the number of integers among the first k positive integers which are coprime with k.

7. 符號$(a_k)_{k\geq 1}$表示一組有序的數列：$a_1,a_2,\cdots a_n,a_{n+1},\cdots$。
   The symbol $(a_k)_{k\geq 1}$ represents an ordered sequence of numbers: $a_1,a_2,\cdots a_n,a_{n+1},\cdots$。

8. $\sum_{k=1}^{n}a_k=a_1+a_2+\cdots+a_n$.　$\sum_{k=1}^{n}a_kb_k=a_1b_1+a_2b_2+\cdots+a_nb_n$.

9. 對任意的整數k,n 使得$0\leq k\leq n$，定義$\binom{n}{k}=\dfrac{n(n-1)\cdots(n-k+1)}{(1)(2)\cdots(k-1)k}$。
   For any integer $0\leq k\leq n$ such that $0\leq k\leq n$, define $\binom{n}{k}=\dfrac{n(n-1)\cdots(n-k+1)}{(1)(2)\cdots(k-1)k}$.

1. 求$\tan(15°)$的值。

Find the value of $\tan(15°)$.

2. 若x, y, z 爲正實數且$xy \neq 1,\ yz \neq 1, zx \neq 1,\ xyz \neq 1$，則
$\dfrac{1}{\log_{xy}(xyz)} + \dfrac{1}{\log_{yz}(xyz)} + \dfrac{1}{\log_{zx}(xyz)} =$ ______________。

If x, y, z are positive numbers such that $xy \neq 1,\ yz \neq 1,\ zx \neq 1$ and $xyz \neq 1$, then
$\dfrac{1}{\log_{xy}(xyz)} + \dfrac{1}{\log_{yz}(xyz)} + \dfrac{1}{\log_{zx}(xyz)} =$ ______________.

3. 若x, y, z 爲正實數滿足$\tan^{-1}(x) + \tan^{-1}(y) + \tan^{-1}(z) = \pi$，則$\frac{1}{xy} + \frac{1}{yz} + \frac{1}{zx} =$ ______________。

If x, y, z are positive numbers such that $\tan^{-1}(x) + \tan^{-1}(y) + \tan^{-1}(z) = \pi$, then
$\dfrac{1}{xy} + \dfrac{1}{yz} + \dfrac{1}{zx} =$ ______________.

4. 記$\varphi(k)$爲歐拉函數在正整數k的取值。求：$\displaystyle\sum_{k=1}^{n} \varphi(k) \left\lfloor \frac{n}{k} \right\rfloor =$ ______________。

Denote by $\varphi(k)$ the value of the Euler-totient function evaluated at k.

Find the value $\displaystyle\sum_{k=1}^{n} \varphi(k) \left\lfloor \frac{n}{k} \right\rfloor =$ ______________.

5. 求$\dfrac{bc}{2a^2 + bc} + \dfrac{ab}{2c^2 + ab} + \dfrac{ac}{2b^2 + ac}$的最小值，其中$a, b, c$ 爲任意正實數。

Find the minimum value of $\dfrac{bc}{2a^2 + bc} + \dfrac{ab}{2c^2 + ab} + \dfrac{ac}{2b^2 + ac}$,
where a, b, c are positive real numbers.

6. 已知$ABCD$是凸四邊形，求證以下不等式

$$\max(AB + CD, AD + BC) < AC + BD < AB + BC + CD + DA。$$

Let $ABCD$ be a convex quadrilateral. Prove that

$$\max(AB + CD, AD + BC) < AC + BD < AB + BC + CD + DA.$$

7. 求最大的實數 k 使得對於任意的正數a和b，均有 $(a + b)(ab + 1)(b + 1) \geq kab^2$，並給出證明。

Determine, with proof, the largest real number k such that $(a + b)(ab + 1)(b + 1) \geq kab^2$ holds for any positive numbers a and b.

8. 設S爲全體形如$a^3 + b^3 + c^3 - 3abc$ 的整數所組成的集合，其中a, b, c爲任意的整數。
求證：若$x, y \in S$，則$xy \in S$。

Let S be the set of all integers in the form of $a^3 + b^3 + c^3 - 3abc$, where a, b and c are integers.
Prove that if $x, y \in S$, then $xy \in S$.

9. 已知在$n \times n$的表格内第i行及第j列填入數a_{ij}，其中$1 \leq i, j \leq n$，且滿足以下條件：任何在不同行與不同列的n個數之和均相等。例如，當$n = 2$時，$a_{11} + a_{22} = a_{12} + a_{21}$。
求證：存在數$x_1, x_2, \cdots, x_n$ 及 $y_1, y_2, \cdots, y_n$使得對任意的整數$1 \leq i, j \leq n$有$a_{ij} = x_i + y_j$。

The entries of an $n \times n$ array of numbers are denoted by $a_{ij}, 1 \leq i, j \leq n$. All the sums of any n entries lying on different rows and different columns are equal. For example, $a_{11} + a_{22} = a_{12} + a_{21}$ if $n = 2$.
Prove that there exist numbers $x_1, x_2, \cdots, x_n$ and $y_1, y_2, \cdots, y_n$, such that $a_{ij} = x_i + y_j$, for all $1 \leq i, j \leq n$.

10. 設n爲正整數，求證：
$$\sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k} = \sum_{k=0}^{n} 2^k \binom{n}{k}^2。$$

Let n be a positive integer. Prove that
$$\sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k} = \sum_{k=0}^{n} 2^k \binom{n}{k}^2.$$

11. 試確定滿足以下三個條件的所有正整數數列$(a_n)_{n \geq 1}$：

(i) 對任意的整數$n \geq 1$都有$a_n < a_{n+1}$；
(ii) 對任意的整數$n \geq 1$都有$a_{2n} = a_n + n$；
(iii) 若a_n是素數，則n是素數。

並給出以上的證明。
Determine, with proof, all sequences $(a_n)_{n \geq 1}$ of positive integers such that

(i) $a_n < a_{n+1}$ for all $n \geq 1$;
(ii) $a_{2n} = a_n + n$ for all $n \geq 1$;
(iii) if a_n is a prime, then n is a prime.

12. 如下圖所示設，H與O分別爲三角形ABC的垂心及外心。試確定向量和$\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} - \overrightarrow{OH}$，其中$\overrightarrow{OA}$、$\overrightarrow{OB}$、$\overrightarrow{OC}$及$\overrightarrow{OH}$爲向量，並給出證明。

As shown in the figure below, let H, O be the orthocenter and the circumcenter of a triangle ABC respectively. Determine, with proof, the sum $\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} - \overrightarrow{OH}$ where $\overrightarrow{OA}, \overrightarrow{OB}, \overrightarrow{OC}$ and $\overrightarrow{OH}$ are vectors.