2022/2023學年學界數學比賽　初中組

Concurso Interescolar de Matemática do ano lectivo de 2022/2023　Divisão Júnior

Macau Interschool Mathematics Competition in 2023/2023　Junior Division

學校 School ______________________ 姓名 Name ______________________ 座號 Seat No.__________

- 比賽共有 16 題：題 1–4 是選擇題，題 5–7 是填空題，每題 5 分，請把題 1–7 的答案填寫在右方空格內。題 $9-16$ 是證明題，每題 10 分，請把詳細的解題過程寫在背後的白紙上，並在左上角寫上題目編號及姓名。禁止使用任何類型之計算機或計算工具。

- There are 16 questions. Questions 1–4 are multiple choice, questions 5–7 are fill-in-the-blanks, and each of them is 5 marks. Fill in the correct answers of Questions 1–7 in the boxes on the right. Questions 9–16 are proof-related, each of them is 10 marks. Please write the detailed arguments and steps of your solutions at the blank page with question number and your name on the left-upper corner. Any electronic calculators or computing tools of any kinds are not allowed.

符號　Notations

1. 符號 $|x|$ 表示實數 x 的絕對值。例如 $|-13|=13,\ |0|=0,\ |13|=13$。
   The symbol $|x|$ represents the absolute value of real number x.
   For example, $|-13|=13,\ |0|=0,\ |13|=13$.

2. 設 a,b 爲整數且 $a\neq 0$, 符號 $a\mid b$ 表示 b 是 a 的倍數, 即有某個整數 q 使得 $b=aq$ 。
   Let a,b be two integers and $a\neq 0$, the notation $a\mid b$ means that b is a multiple of a, i.e. $b=aq$ for some integer q.

3. $N=\left\lfloor\frac{a}{k}\right\rfloor$ 代表分數 $\frac{a}{k}$ 的整數部份, 即 N 是滿足不等式 $n\leq\frac{a}{k}$ 的所有整數 n 中的最大值。
   $N=\left\lfloor\frac{a}{k}\right\rfloor$ represents the integral part of the fraction $\frac{a}{k}$, that is, among all integers n satisfying the inequality $n\leq\frac{a}{k}$, N is the largest one.

4. 設 k 爲整數，稱 a 爲 k 的正約數 若 a 滿足以下兩個條件：(i) a 是正整數及 (ii) k 是 a 的倍數。
   Let k be an integer, a is called a positive divisor of k, if a satisfies the following two conditions: (i) a is a positive integer and (ii) k is a multiple of a.

5. 符號 $\gcd(a,\ b,\ c,\ d)$ 代表整數 a,b,c,d 的最大公約數。
   The symbol $\gcd(a,\ b,\ c,\ d)$ represents the the greastest common divisor of integers a,b,c and d.

6. 符號 $\sigma(k)$ 代表正整數 k 的所有正約數之和。
   The symbol $\sigma(k)$ represents the sum of all positive divisors of positive integer k.

7. $\sum_{k=1}^{n}a_k=a_1+a_2+\cdots+a_n;\qquad \sum_{k=1}^{n}a_kb_k=a_1b_1+a_2b_2+\cdots+a_nb_n.$

8. 符號 S_{ABC} 代表三角形 ABC 的面積。
   The symbol S_{ABC} represents the area of triangle ABC.

9. 正整數 p 稱爲素數(或者質數)，如果 $p>1$ 且 p 恰有兩個正約數 1 及 p。
   A positive integer p is called a prime number, if $p>1$ and p has only 2 positive divisors 1 and p.

1. 2023 共有__________個正約數。

The number of positive divisors of 2023 is __________.

A. 1 **B.** 4 **C.** 6 **D.** 12 **E.** 以上皆非 None of the above

2. 若實數 $a,\ b,\ c$ 互不相等，問：$\frac{a-b}{b-c},\ \frac{b-c}{c-a},\ \frac{c-a}{a-b}$ 中有多少個負數?

If a, b, c are distinct real numbers, how many of negative numbers among $\frac{a-b}{b-c}, \frac{b-c}{c-a}, \frac{c-a}{a-b}$ are there?

A. 0 **B.** 1 **C.** 2 **D.** 3 **E.** 以上皆非 None of the above

3. 由小至大排列 $p = 3^{70},\ q = 17^{28},\ r = 6^{42}$。

Arrange $p = 3^{70},\ q = 17^{28},\ r = 6^{42}$ in ascending order.

A. $p<q<r$ **B.** $q<p<r$ **C.** $r<p<q$ **D.** $p<r<q$ **E.** $q<r<p$

4. 與 2023 最接近的完全立方數是__________。

The closest perfect cube to 2023 is __________.

A. 1729 **B.** 1971 **C.** 2020 **D.** 2197 **E.** 以上皆非 None of the above

5. 若實數 x、y 滿足 $(x+y)^2 = 2025$ 且 $(x-y)^2 = 289$，則 $x^2 - xy + y^2 =$______________。

If x, y are real numbers satisfying $(x+y)^2 = 2025$ and $(x-y)^2 = 289$, then $x^2 - xy + y^2 =$ ______________.

6. 已知 $\left(1+\frac{1}{x}\right)\left(1+\frac{1}{y}\right)\left(1+\frac{1}{z}\right) = 3$ 且 $x<y<z$，則滿足以上條件的正整數解組 (x, y, z) 的個數是______________。

The number of triples (x, y, z) of positive integers satisfying $x<y<z$ and $\left(1+\frac{1}{x}\right)\left(1+\frac{1}{y}\right)\left(1+\frac{1}{z}\right) = 3$ is ______________.

7. 設 x 爲任意的實數，則 $\sqrt{x^2+2x+5} + \sqrt{x^2-6x+34}$ 的最小值是______________。

If x is any real number, then the minimum value of $\sqrt{x^2+2x+5} + \sqrt{x^2-6x+34}$ is ______________.

8. 設 ABC 爲直角三角形，其中 $\angle BAC = 90°$ 及 $\angle ABC = 30°$，$\angle B$ 的平分線交直線 AC 於點 P。若 $AB = \sqrt{3}$，求 AP 的長度，並給出證明。

Let ABC be a right-angled triangle with $\angle BAC = 90°, \angle ABC = 30°$. The angle bisector of $\angle B$ meets line AC at point P. If $AB = \sqrt{3}$, determine, with proof, the length of AP.

9. 設 M 爲凸四邊形 $ABCD$ 內的一點使得 $S_{MAB} = S_{MBC} = S_{MCD} = S_{MDA}$。求證：$ABCD$ 內的某條對角線經過另一條對角線的中點。

Let M a point inside a convex quadrilateral $ABCD$ such that $S_{MAB} = S_{MBC} = S_{MCD} = S_{MDA}$. Prove that one of the diagonals of $ABCD$ passes through the midpoint of the other diagonal.

10. 假設 $p,\ p+d,\ p+2d,\ p+3d,\ p+4d$ 和 $p+5d$ 是六個素數，其中 p 和 d 是正整數。證明 d 是 30 的倍數。

Suppose that $p, p+d, p+2d, p+3d, p+4d$, and $p+5d$ are six primes, where p and d are positive integers. Show that d is a multiple of 30.

11. 記 $\sigma(k)$ 爲正整數 k 的所有正約數之和。設 n 爲正整數，求證：$\sum_{k=1}^{n} \sigma(k) = \sum_{k=1}^{n} k\left\lfloor \frac{n}{k} \right\rfloor$。

Denote by $\sigma(k)$ the sum of all positive divisors of positive integer k.

For any positive integer n, prove that $\sum_{k=1}^{n} \sigma(k) = \sum_{k=1}^{n} k\left\lfloor \frac{n}{k} \right\rfloor$.

12. 設 a 爲正實數且滿足 $a^3 = 6(a+1)$。求二次方程 $x^2 + ax + a^2 - 6 = 0$ 實根的個數，並給出證明。

Let a be positive real number such that $a^3 = 6(a+1)$.

Determine, with proof, the number of real roots of the equation $x^2 + ax + a^2 - 6 = 0$.

13. 求以下函數 $f(x) = |x-1| + |x-2| + \cdots + |x-100|$ 的最小值，其中 x 是任意的實數，並給出證明。

Determine, with proof, the minimum value of function $f(x) = |x-1| + |x-2| + \cdots + |x-100|$, where x is any real number.

14. 試找出滿足 $\gcd(a,b,c,d) = 1,\ a \mid (b+c),\ b \mid (c+d),\ \mid (d+a),\ d \mid (a+b)$ 的所有正整數四元組 (a, b, c, d)，並給出證明。

Determine, with proof, all quadruples (a, b, c, d) of positive integers satisfying

$$\gcd(a,b,c,d) = 1, \quad a \mid (b+c), \quad b \mid (c+d), \quad c \mid (d+a), \quad d \mid (a+b).$$

15. 實數 $a,\ b,\ c$ 滿足 $(a+b+c)\left(\frac{1}{a+b-5c} + \frac{1}{b+c-5a} + \frac{1}{c+a-5b}\right) = \frac{9}{5}$，求 $(a+b+c)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)$ 的值，並給出證明。

If a, b and c are real numbers such that $(a+b+c)\left(\frac{1}{a+b-5c} + \frac{1}{b+c-5a} + \frac{1}{c+a-5b}\right) = \frac{9}{5}$,

Determine, with proof, the value of $(a+b+c)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)$.

16. 設 n 爲正整數，求證：$\left\lfloor \left(\frac{1+\sqrt{5}}{2}\right)^{4n-2} \right\rfloor - 1$ 是完全平方數。

Let n be a positive integer. Prove that $\left\lfloor \left(\frac{1+\sqrt{5}}{2}\right)^{4n-2} \right\rfloor - 1$ is a perfect square.