姓名 Name____________________________________(in ID card)

學校 School____________________________________(No short form)

班級 Form__________________________

座位編號 Seat Number__________________________

此卷有 10 道題目：題 1 – 3 每題 8 分; 題 4 – 6 每題 10 分; 題 7 – 10 每題 12 分。
There are 10 questions in this paper: Questions 1 – 3 have 8 marks each;
Questions 4 – 6 have 10 marks each; Questions 7 – 10 have 12 marks each.

可用鉛筆、黑色或藍色的筆填寫。
You can write with pencil, black or blue pens.

手機號 Phone No__________________________
(可以不填手機號, 只為通知有關訓練及測試.
not necessary to fill in phone no, just for passing information of training and test)

未有通知前不能翻閱試卷

Don't FLIP this paper without consent.

2021 年高中組　　　2021 High School Senior Division

請把題目的解答寫在空白的 **A4** 紙上, 需要時可向工作人員索取更多的白紙.

Write your answer on blank $A4$ papers. Request more papers if needed.

1. 簡化以下的和 $1+\sum_{n=1}^{2021}(-1)^n\dfrac{n^2+n+1}{n!}$.

   Simplify the sum $1+\sum_{n=1}^{2021}(-1)^n\dfrac{n^2+n+1}{n!}$.

2. 記 $P(x)=x^2+ax+b$, 其中 $a,b\in[-2,2]$. 當 a,b 在閉區間 $[-2,2]$ 上變動時, 求二次方程 $P(x)=0$ 的實數解的取值範圍.

   Let $P(x)=x^2+ax+b$, where $a,b\in[-2,2]$. If a,b vary within the closed interval $[-2,2]$, determine the range (set) of all real roots of quadratic equation $P(x)=0$.

3. 在 xy 平面上, 記 $f(a,b)$ 為點 (a,b) 到直線 $\ell: 3x+4y=1$ 的距離. 當 a,b 走遍所有整數且 (a,b) 不在直線 ℓ, 確定 $f(a,b)$ 的最小值.

   On xy-plane, denote by $f(a,b)$ the distance from the point (a,b) to the line $\ell: 3x+4y=1$. If a and b vary among all integers, and (a,b) is not on the line ℓ, determine the minimum value of $f(a,b)$.

4. 將邊長為 n 的等邊三角形的每條邊等分 n 份, 然後用平行這三邊的線段連起這些等分點, 得出一些由邊長為 1 的等邊三角形所組成的圖形 S_n. 設圖形 S_n 共有 N_n 個邊長 1 至 n 的等邊三角形. 下圖給出當 $n=1,2,3$ 的圖形, $N_1=1, N_2=5$. 試找出用 n 表示 N_n 的公式.

$n=1$　　$n=2$　　$n=3$

   Dividing each of the 3 sides of an equilateral triangle of length n into n equal parts, and using segments parallel to the 3 sides to join the division points, one obtains a figure S_n consists of some equilateral triangles of length 1. Suppose that S_n contains N_n equilateral triangles of lengths 1 through n. The figures above show S_1, S_2, S_3, and $N_1=1, N_2=5$. Express N_n in terms of n.

5. (i) 設 x,y,z 為正數，求證 $x^{x-y}y^{y-z}z^{z-x}\ge 1$.

   Let x,y and z be positive, prove that $x^{x-y}y^{y-z}z^{z-x}\ge 1$.

   (ii) 設 a,b,c 為正數，求證 $a^4+b^4+c^4\ge a^2bc+ab^2c+abc^2$.

   Let a,b and c be positive, prove that $a^4+b^4+c^4\ge a^2bc+ab^2c+abc^2$.

6. 定義 $f(x)=\dfrac{9^x}{9^x+3}$. 計算 $f(\frac{1}{2021})+f(\frac{2}{2021})+\cdots+f(\frac{2020}{2021})$.

   Define $f(x)=\dfrac{9^x}{9^x+3}$. Evaluate the sum $f(\frac{1}{2021})+f(\frac{2}{2021})+\cdots+f(\frac{2020}{2021})$.

7. 記 $[x]$ 為小於或等於 x 的最大整數, 如 $[\pi]=3$. 定義數列 $a_1,a_2,\cdots$ 如下:

   $$\text{當 } n\ge 1,\quad a_n=\frac{1}{n}\left(\left[\frac{n}{1}\right]+\left[\frac{n}{2}\right]+\cdots+\left[\frac{n}{n}\right]\right).$$

   (a) 求證: 存在無窮多個整數 $n\ge 1$ 使得 $a_{n+1}>a_n$;

   (b) 確定並證明: 是否存在無窮多個整數 $n\ge 1$ 使得 $a_{n+1}<a_n$.

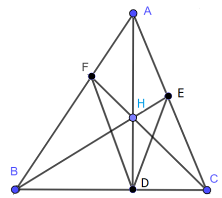
   Denote by $[x]$ the largest integer which is less than or equal to x, for example $[\pi]=3$. Define $a_n=\frac{1}{n}\left(\left[\frac{n}{1}\right]+\left[\frac{n}{2}\right]+\cdots+\left[\frac{n}{n}\right]\right)$ for all $n\ge 1$.

   (a) Prove that there exist infinitely many integers $n\ge 1$ such that $a_{n+1}>a_n$;

   (b) Determine with proof if there are infinitely many integers $n\ge 1$ such that $a_{n+1}<a_n$.

8. 如圖在銳角三角形 ABC 中, AD 是邊 BC 上的高, H 是線段 AD 內任一點, BH 和 CH 的延長線分別交 AC 和 AB 於 E 和 F, 求證 $\angle EDH=\angle FDH$.

   As shown in the figure, ABC is an acute triangle, AD is the height from BC, and H is a point on the segment AD. If the lines BH and CH meet AC and AB at points E and F respectively. Prove that $\angle EDH=\angle FDH$.



9. 確定實係數多項式 $P(x)$ 使得多項式 $(x+1)P(x-1)-(x-1)P(x)$ 只有常數項.

   Determine, with proof, all polynomials $P(x)$ with real coefficients such that the following polynomial $(x+1)P(x-1)-(x-1)P(x)$ is a constant.

10. 已知在 $\triangle ABC$, $\angle A<\angle B<90^\circ$. 圓 Γ 過點 A,B,C. 圓 Γ 過點 A,C 的兩條切線交於 P. 直線 AB 與 PC 交於 Q. 若三角形 ACP, ABC, BQC 有相同的面積, 求證 $\angle BCA=90^\circ$.

    Let ABC be a triangle with $\angle A<\angle B<90^\circ$, and let Γ be the circle through A,B,C. The tangents to Γ at A and C meet at P. The lines AB and PC meet at Q. If the areas of triangles ACP, ABC, BQC are equal, prove that $\angle BCA=90^\circ$.