2020 高中組 High-School Senior Division

姓名 Name________________________________(in ID card)

學校 School________________________________(No short form)

班級 Form______________________

座位編號 Seat Number______________________

此卷有 16 道題目：

I. 題 1 – 5 是證明題，每題 15 分，必須填寫完整的計算或證明。

II. 題 6 – 9 道是選擇題，每題 5 分，只須在方格 填寫英文字母：**A,B,C, D, E**。

III. 題 10 – 16 是填空題，每題 8 分，只須填寫正確答案，不須填寫過程。

There are 16 questions in this paper:

I. The questions 1 – 5 are long questions requiring detailed proofs, 15 marks each.

II. The questions 6 – 9 are multiple-choice, 5 marks each.
Fill in **A, B, C, D, E** in the boxes provided.

III. The questions 10 – 16 are fill-in-blanks, 8 marks each. Fill in final answers, and no steps are needed.

可用鉛筆、黑色或藍色的筆填寫。

You can write with pencil, black or blue pens.

手機號 Phone No______________________

(可以不填手機號, 只為通知有關訓練及測試.
not necessary to fill in phone no, just for passing information of training and test)

I 長題目每題 15 分 Long Questions 15 marks each

1. 5 位朋友各有一枚金幣，他們同時把自己唯一的一枚金幣送予其他四位朋友之一，當完成交換金幣後這稱之為一種操作。問：有多少種不同的操作使得操作後這 5 位朋友各仍有一枚金幣?

Each of 5 friends has a gold coin. Everyone gives his own gold coin to one of his 4 friends at the same time. After they have completed exchange the gold coins, this is called an *operation*. Determine the number of different operations in which each of them finds out that he still has a gold coin in his hand.

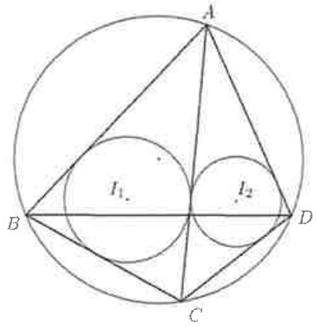
2. 定義數列 $\{x_n\}_{n\geq 1}$ 如下：$x_1 = 5$ 及 $x_{k+1} = x_k^2 - 3x_k + 3$ 當 $k = 1, 2, 3, \cdots$。
求證：對任意 $k \geq 0$，有 $x_{k+1} \geq 3^{2^k}$。
Define a sequence $\{x_n\}_{n\geq 1}$ by $x_1 = 5$, and $x_{k+1} = x_k^2 - 3x_k + 3$ for $k = 1, 2, 3, \cdots$.
Prove that $x_{k+1} \geq 3^{2^k}$ for any $k \geq 0$.

3. 設 a, b, c 為正數且滿足 $a + b + c = 2$。求證：

$$\frac{(a-1)^2}{b} + \frac{(b-1)^2}{c} + \frac{(c-1)^2}{a} \geq \frac{1}{4}\left(\frac{a^2+b^2}{a+b} + \frac{b^2+c^2}{b+c} + \frac{c^2+a^2}{c+a}\right).$$

Let a, b, c be positive numbers such that $a + b + c = 2$. Prove the inequality above.

4. 如右圖所示 $ABCD$ 為圓內接四邊形。$\odot I_1$ 和 $\odot I_2$ 分別是 $\triangle ABC$ 和 $\triangle ADC$ 的內切圓。
若 $\angle BAC = \angle DAC$，求證 $\odot I_1$ 與 $\odot I_2$ 的某一條外公切線與 BD 平行。
As shown in the right figure, $ABCD$ is concyclic, $\odot I_1$ and $\odot I_2$ are incircles of $\triangle ABC$ and $\triangle ACD$ respectively. If $\angle BAC = \angle DAC$, prove that one of the external common tangents of $\odot I_1$ and $\odot I_2$ is parallel to BD.



5. (a) 試求四個連續正整數使得最大數的立方是其餘三個數的立方之和。
Find 4 consecutive positive integers such that the cube of the largest number is equal to the sum of the cubes of remaining 3 numbers.

(b) 求滿足 $x^3 + y^3 + 3xy = 1$ 的所有整數對 (x, y)。
Determine all pairs (x, y) of integers such that $x^3 + y^3 + 3xy = 1$.

II 選擇題 每題 5 分 Multiple Choice Questions 5 marks each

6. $N = 2020^2 - 2019^2 + 2018^2 - \cdots + 2^2 - 1^2$.
   A. $N < 0$ **B.** $N = 0$ **C.** $0 < N < 2020$
   D. $2020 \le N < 4040$ **E.** 以上皆非 None of the above.

7. 若 A, B, C, D 是實數使得 $\frac{6x^3+10x}{x^4+x^2+1} \equiv \frac{Ax+B}{x^2+x+1} + \frac{Cx+D}{x^2-x+1}$ 是恆等式，求 $A+B+3C+3D$。
   If A, B, C, D are real numbers such that $\frac{6x^3+10x}{x^4+x^2+1} \equiv \frac{Ax+B}{x^2+x+1} + \frac{Cx+D}{x^2-x+1}$ is an identity, find $A+B+3C+3D$.
   A. 2 **B.** 4 **C.** 8 **D.** 16 **E.** 以上皆非 None of the above.

8. 正實數 a、b、c 滿足 $abc = 1$，求 $\frac{1}{ab+a+1} + \frac{1}{bc+b+1} + \frac{1}{ac+c+1}$。
   Positive real numbers a, b and c satisfy $abc = 1$, find $\frac{1}{ab+a+1} + \frac{1}{bc+b+1} + \frac{1}{ac+c+1}$.
   A. -1 **B.** 0 **C.** 1 **D.** 2 **E.** 以上皆非 None of the above.

9. $(a+b+c)^3 + (b-a-c)^3 + (c-a-b)^3 + (a-b-c)^3 =$
   A. $-24abc$ **B.** $-12abc$ **C.** $12abc$ **D.** $24abc$
   E. 以上皆非 None of the above.

III 填空題 每題 8 分 Fill-in-blank Questions 8 marks each

10. 試求 $f(x) = x^4 - 4x$ 的最小值 M，其中 x 是任意的實數。
    Determine the minimum value M of $f(x) = x^4 - 4x$, where x is any real number.

    $M =$

11. 記 $2^{2020} \times 5^{2019} \times 9^{2018} = \cdots a_{2020}a_{2019}a_{2018}\cdots a_1a_0$ 為十進制表示，其中數碼 a_i 為 $0, 1, \cdots, 8, 9$ 之一。求 $a_{2020} + a_{2021}$。
    Write $2^{2020} \times 5^{2019} \times 9^{2018} = \cdots a_{2020}a_{2019}a_{2018}\cdots a_1a_0$ in decimal format, where i-th digit a_i is one of $0, 1, \cdots, 8, 9$. Find $a_{2020} + a_{2021}$.

    $a_{2020} + a_{2021} =$

12. (a) 設 $f(x) = x^3 - 3abx + a^3 + b^3$，簡化 $f(-a-b)$。
    Simplify $f(-a-b)$, where $f(x) = x^3 - 3abx + a^3 + b^3$.
    (b) 因式分解 $a^3 + b^3 + c^3 - 3abc$。 Factorize $a^3 + b^3 + c^3 - 3abc$.

    (a) $f(-a-b) =$

    (b) $a^3 + b^3 + c^3 - 3abc =$

13. 把 2020 寫成五個 (不一定互異) 整數的立方之和。
    Express 2020 as a sum of the cubes of 5 (not necessarily distinct) integers.

    $2020 =$

14. 試求滿足 $ab^3 = -135$ 及 $(a+b)b = -6$ 的所有實數對 (a, b)。
    Find all pairs (a, b) of real numbers such that $ab^3 = -135$ and $(a+b)b = -6$.

    $(a, b) =$

15. 設 $M(b, c)$ 為函數 $|2x^2 + bx + c|$ 在閉區間 $[0, 2]$ 上的最大值。試求 $M(b, c)$ 的最小值 m ，其中 b, c 走遍所有實數。
    Let $M(b, c)$ be the maximum value of function $|2x^2 + bx + c|$ on the closed interval $[0, 2]$. Find the minimum value m of $M(b, c)$ for all real numbers b and c.

    $m =$

16. 令 $f(n) = \frac{1}{2^n} + \frac{1}{3^n} + \cdots + \frac{1}{2020^n}$，求 $f(2) + f(3) + f(4) + \cdots$ 的值 S。
    Let $f(n) = \frac{1}{2^n} + \frac{1}{3^n} + \cdots + \frac{1}{2020^n}$. Find the value of $S = f(2) + f(3) + f(4) + \cdots$.

    $S =$